Extended Fourier analysis of signals

A Fourier transform is a powerful tool of signal analysis and representation of a time function $x(t)$ (hereinafter referred to as the signal) in the frequency domain

$$F(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt\,, \tag{1.1}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,e^{j\omega t}\,d\omega\,. \tag{1.2}$$

The orthogonality property of the Fourier transform

$$\int_{-\infty}^{\infty} e^{j\omega_0 t}e^{-j\omega t}dt = 2\pi\delta(\omega-\omega_0) \tag{2}$$

providing a basis for the signal selective frequency analysis, where $\delta(\omega-\omega_0)$ is the Dirac delta function. Unfortunately, the Fourier transform calculation according to (1.1) requiring knowledge of the signal $x(t)$ as well as performing of integration operation in infinite time interval. Therefore, for practical evaluation of (1.1) numerically, the signal observation period and the interval of integration is always limited by some finite value Θ, $-\Theta/2 \le t \le \Theta/2$. The same applies to the Fourier analysis of the signal $x(t)$ sampled versions: nonuniformly sampled signal $x(t_k)$ or uniformly sampled signal $x(kT)$, $k = -\infty,\ldots,-1,0,1,\ldots,+\infty$. Only a finite length sequence $x(t_k)$ or $x(kT)$, $k=0,1,2,\ldots,K-1$, are subject of Fourier analysis, where K is a discrete sequence length, T is sampling period and the signal observation period $\Theta=t_{K-1}-t_0$ or $\Theta=KT$. To satisfy the Nyquist limit, uniform sampling of continuous time signal should be performed with the sampling period $T\le\pi/\Omega$, where Ω is upper cyclic frequency of signal $x(t)$. Although nonuniform sampling has no such strict limitation on the mean sampling period $T_s=\Theta/K$, the following analysis we suppose that both sequences, $x(t_k)$ and $x(kT)$, are derived from the band-limited in Ω signal $x(t)$. Let write the basic expressions of the classical and the proposed extended Fourier analysis of continuous time signal $x(t)$ and its sampled versions $x(t_k)$ and $x(kT)$.

Basic expressions of classical Fourier analysis

The classical Fourier analysis dealing with the following finite time Fourier transforms:

$$F_\Theta(\omega) = \int_{-\Theta/2}^{\Theta/2} x(t)e^{-j\omega t} dt\,, \tag{3.1}$$

$$F_\Theta(\omega) = \sum_{k=0}^{K-1} x(t_k)e^{-j\omega t_k}\,, \tag{3.2}$$

$$F_\Theta(\omega) = \sum_{k=0}^{K-1} x(kT)e^{-j\omega kT}\,, \tag{3.3}$$

$$x_\Theta(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} F_\Theta(\omega)\,e^{j\omega t}\,d\omega\,, \tag{3.4}$$

where (3.4) is the inverse Fourier transform obtained from (1.2) for band-limited in Ω signal. Transforms (3.2 and 3.3) are known as Discrete Time Fourier Transforms (DTFT) of nonuniformly and uniformly sampled signals. The signal amplitude spectrum is the Fourier transforms (3.1-3.3) results, divided by the observation period Θ,

$$S_\Theta(\omega) = \frac{1}{\Theta} F_\Theta(\omega). \tag{4}$$

The frequency resolution of the classical Fourier analysis is inversely proportional to the signal observation period Θ.
Obviously, one can get the formula (3.1) by truncation of infinite integration limits in (1.1) and the DTFT (3.2) and (3.3) as result of replacement of infinite sums by finite ones. This mean, the classical Fourier analysis supposed that the signal outside Θ is zeros. In other words, the Fourier transform calculation by formulas (3.1-3.3) is well justified if applied to time-limited within Θ signals. On the other hand, a band-limited in Ω signal can not be also time-limited and obviously have nonzero values outside Θ. Generally, the Fourier analysis results obtained by using the exponential basis $e^{j\omega t}$, $e^{-j\omega t_k}$ and $e^{-j\omega kT}$ tend to the Fourier transform (1.1), if $\Theta \to \infty$, while in any finite Θ there may exist another transform basis functions providing a more accurate estimation of (1.1).

Basic expressions of extended Fourier analysis

The idea of extended Fourier analysis is finding the transform basis functions, applicable for a band-limited signals registered in finite time interval Θ and providing the results as close as possible to the Fourier transform (1.1) defined in infinite time interval. The formulas for proposed extended Fourier analysis could be written as

$$F_\alpha(\omega) = \int_{-\Theta/2}^{\Theta/2} x(t)\alpha(\omega,t)dt, \tag{5.1}$$

$$F_\alpha(\omega) = \sum_{k=0}^{K-1} x(t_k)\alpha(\omega,t_k), \tag{5.2}$$

$$F_\alpha(\omega) = \sum_{k=0}^{K-1} x(kT)\alpha(\omega,kT), \tag{5.3}$$

$$x_\alpha(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} F_\alpha(\omega)e^{j\omega t}d\omega, \tag{5.4}$$

where, in general, the transform basis $\alpha(\omega,t)$, $\alpha(\omega,t_k)$ and $\alpha(\omega,kT)$ are not equal to the classical ones (3.1-3.3). Note that the inverse Fourier transform (5.4) still holds the exponential basis $e^{j\omega t}$.
To ensure that the transforms (5.1-5.3) results will be close to the signal $x(t)$ Fourier transform (1.1) results, the following minimum least square expression will be composed and solved

$$\left|F(\omega) - F_\alpha(\omega)\right|^2 \to \min. \tag{6}$$

Unfortunately, as already stated above, the calculation of $F(\omega)$ for a band-limited signals can not be performed directly. So, in order to compose (6), we should find an adequate substitution. Let remember, that a complex exponent (known as an analytic signal), at cyclic frequency ω_0 and with a complex amplitude $S(\omega_0)$, is defined in infinite time interval

$$x(\omega_0,t) = S(\omega_0)e^{j\omega_0 t}, -\infty < t < \infty. \tag{7}$$

The Fourier transform of an analytic signal can be expressed by the Dirac delta function (2)

$$\int_{-\infty}^{\infty} x(\omega_0,t)e^{-j\omega t}dt = 2\pi S(\omega_0)\delta(\omega-\omega_0). \tag{8}$$

Now, let use (7) as a signal model with known amplitude spectrum $S(\omega_0)$ for frequencies in range $-\Omega \le \omega_0 \le \Omega$ and, in the minimum least square expression (6), substitute $F(\omega)$ by the signal model Fourier transform (8) and the signals $x(t)$, $x(t_k)$ and $x(kT)$ in (5.1-5.3) by the signal models (7), correspondingly. Finally, the minimum least square error estimators for all the three signal cases get the following form

$$\Delta = \int_{-\Omega}^{\Omega} \left| 2\pi S(\omega_0)\delta(\omega-\omega_0) - \int_{-\Theta/2}^{\Theta/2} S(\omega_0)\, e^{j\omega_0 t}\, \alpha(\omega,t)dt \right|^2 d\omega_0 , \quad (9.1)$$

$$\Delta = \int_{-\Omega}^{\Omega} \left| 2\pi S(\omega_0)\delta(\omega-\omega_0) - \sum_{k=0}^{K-1} S(\omega_0) e^{j\omega_0 t_k} \alpha(\omega,t_k) \right|^2 d\omega_0 , \quad (9.2)$$

$$\Delta = \int_{-\Omega}^{\Omega} \left| 2\pi S(\omega_0)\delta(\omega-\omega_0) - \sum_{k=0}^{K-1} S(\omega_0) e^{j\omega_0 kT} \alpha(\omega,kT) \right|^2 d\omega_0 . \quad (9.3)$$

The solution of (9.1-9.3) for a definite signal model (7) provide the basis functions $\alpha(\omega,t)$, $\alpha(\omega,t_k)$ and $\alpha(\omega,kT)$ for extended Fourier transforms (5.1-5.3). To control how close the selected signal model amplitudes $S(\omega_0)$ are to the signals $x(t)$, $x(t_k)$ and $x(kT)$ amplitude spectrum, we will find the formulas for estimate amplitude spectrum $S_\alpha(\omega)$ in the extended Fourier basis $\alpha(\omega,t)$, $\alpha(\omega,t_k)$ and $\alpha(\omega,kT)$. The formula (8) is showing the connection between the signal model Fourier transform and its amplitude spectrum $S(\omega_0)$. Amplitude spectrum $S_\alpha(\omega)$ are calculated as transforms (5.1-5.3) divided by estimation of $2\pi\delta(\omega)$ in extended Fourier basis, obtained from integrands of Δ estimators (9.1-9.3) for $\omega_0=\omega$,

$$S_\alpha(\omega) = \frac{\int_{-\Theta/2}^{\Theta/2} x(t)\alpha(\omega,t)dt}{\int_{-\Theta/2}^{\Theta/2} e^{j\omega t}\alpha(\omega,t)dt} , \quad (10.1)$$

$$S_\alpha(\omega) = \frac{\sum_{k=0}^{K-1} x(t_k)\alpha(\omega,t_k)}{\sum_{k=0}^{K-1} e^{j\omega t_k}\alpha(\omega,t_k)} , \quad (10.2)$$

$$S_\alpha(\omega) = \frac{\sum_{k=0}^{K-1} x(kT)\alpha(\omega,kT)}{\sum_{k=0}^{K-1} e^{j\omega kT}\alpha(\omega,kT)} . \quad (10.3)$$

Values of the denominator in formulas (10.1-10.3) are in inverse ratio to the frequency resolution of the extended Fourier transform. For example, after substituting exponential basis $\alpha(\omega,t) = e^{-j\omega t}$ in (10.1), the denominator becomes equal to Θ as in formula (4) for the classical Fourier analysis. To establish relationships between classical and extended Fourier analysis, let consider a special case of Δ estimators (9.1-9.3) for the signal model having a rectangular form of amplitude spectrum, $S(\omega_0)=1$ for $-\Omega \le \omega_0 \le \Omega$ and zeros outside.

Extended Fourier analysis: a particular solution

The minimum least square error estimators (9.1-9.3) for the signal model $S(\omega_0)=1$, $-\Omega \le \omega_0 \le \Omega$ and zeros outside, reduces to

$$\Delta = \int_{-\Omega}^{\Omega} \left| 2\pi\delta(\omega-\omega_0) - \int_{-\Theta/2}^{\Theta/2} e^{j\omega_0 t}\alpha(\omega,t)dt \right|^2 d\omega_0 , \tag{11.1}$$

$$\Delta = \int_{-\Omega}^{\Omega} \left| 2\pi\delta(\omega-\omega_0) - \sum_{k=0}^{K-1} e^{j\omega_0 t_k}\alpha(\omega,t_k) \right|^2 d\omega_0 , \tag{11.2}$$

$$\Delta = \int_{-\Omega}^{\Omega} \left| 2\pi\delta(\omega-\omega_0) - \sum_{k=0}^{K-1} e^{j\omega_0 kT}\alpha(\omega,kT) \right|^2 d\omega_0 . \tag{11.3}$$

The solution of (11.1) for continuous time signal $x(t)$ is found as a partial derivation $\frac{\partial\Delta}{\partial\alpha(\omega,\tau)} = 0,\ -\Theta/2 \le \tau \le \Theta/2$, and leads to the linear integral equation

$$\int_{-\Theta/2}^{\Theta/2} \frac{\sin(\Omega(t-\tau))}{\pi(t-\tau)}\alpha(\omega,t)dt = e^{-j\omega\tau} . \tag{12}$$

Step by step solution of (12) is given in **[1]** and **[5]**. Finally, the basis functions $\alpha(\omega,t)$ are found by applying a specific functions system - a prolate spheroidal wave functions $\psi_k(t)$, k=0,1,2,... and are written as series expansion

$$\alpha(\omega,t) = \sum_{k=0}^{\infty} \frac{B_k(\omega)}{\lambda_k}\psi_k(t) . \tag{13}$$

The extended Fourier Transform of continuous time signal $x(t)$ are given by

$$F_\alpha(\omega) = \sum_{k=0}^{\infty} B_k(\omega)a_k ,\ -\Omega \le \omega \le \Omega , \tag{14.1}$$

$$x_\alpha(t) = \sum_{k=0}^{\infty} \psi_k(t)a_k ,\ -\infty < t < \infty , \tag{14.2}$$

$$S_\alpha(\omega) = \frac{\sum_{k=0}^{\infty} B_k(\omega)a_k}{\sum_{k=0}^{\infty} |B_k(\omega)|^2} , \tag{14.3}$$

where $a_k = \frac{1}{\lambda_k}\int_{-\Theta/2}^{\Theta/2} x(\tau)\psi_k(\tau)d\tau$, $\lambda_k = \int_{-\Theta/2}^{\Theta/2} \psi_k^2(t)dt$, $B_k(\omega) = \sqrt{\frac{\pi\Theta}{\lambda_k\Omega}}\psi_k\left(\omega\frac{\Theta}{2\Omega}\right)(-j)^k$.

The extended Fourier transform in accordance with (14.1) requesting a calculations of infinite sums, this mean, an infinite quantity of mathematical operations, therefore it's impossible for real world applications. Theoretically, the value of denominator $\sum_{k=0}^{K} |B_k(\omega)|^2$ in amplitude spectrum formula (14.3) tends to infinite for $K\to\infty$, and the extended Fourier transform (14.1) provide a supper-resolution - an ability to determine the Fourier transform for the sum of complex exponents (sinusoids), if frequencies of them differ by arbitrary small finite value.

The detailed solution steps for the minimum least square error estimators (11.2) and (11.3) are given in articles **[2]** and **[3]**. Similarly to (11.1), finding of the partial derivations $\frac{\partial\Delta}{\partial\alpha(\omega,t_l)} = 0$ and $\frac{\partial\Delta}{\partial\alpha(\omega,lT)} = 0$, for $l = 0,1,2,...,K-1$, leads to the system of linear equations

$$\sum_{k=0}^{K-1} \frac{\sin(\Omega(t_k-t_l))}{\pi(t_k-t_l)}\alpha(\omega,t_k) = e^{-j\omega t_l} , \tag{15.1}$$

$$\sum_{k=0}^{K-1} \frac{\sin(\Omega T(k-l))}{\pi T(k-l)} \alpha(\omega, kT) = e^{-j\omega lT} . \tag{15.2}$$

The solution of (15) expressed in matrix form is

$$\mathbf{A}_\omega = \mathbf{R}^{-1}\mathbf{E}_\omega , \tag{16}$$

where $\mathbf{A}_\omega\,(K\text{x}1)$ and $\mathbf{E}_\omega\,(K\text{x}1)$ are the extended Fourier and the exponential basis functions. The formulas of the Extended Discrete Time Fourier Transform (EDTFT) for signal model $S(\omega_0)$=1, $-\Omega\leq\omega_0\leq\Omega$, are derived by substituting of transform basis (16) into expressions (5) and (10)

$$F_\alpha(\omega) = \mathbf{XR}^{-1}\mathbf{E}_\omega,\ -\Omega \leq \omega \leq \Omega , \tag{17.1}$$

$$x_\alpha(t) = \mathbf{XR}^{-1}\mathbf{E}_\text{t},\ -\infty < t < \infty , \tag{17.2}$$

$$S_\alpha(\omega) = \frac{\mathbf{XR}^{-1}\mathbf{E}_\omega}{\mathbf{E}_\omega^H\mathbf{R}^{-1}\mathbf{E}_\omega} , \tag{17.3}$$

where $\left(^{-1}\right)$ denote the inverse matrix and $\left(^{H}\right)$ denotes Hermitian transpose. The matrices for nonuniformly sampled signal case are composed as follows

$\mathbf{X}\,(1\text{x}K)$ - $x(t_k)$, $\mathbf{E}_\omega\,(K\text{x}1)$ - $e^{-j\omega t_l}$, $\mathbf{R}\,(K\text{x}K)$ - $r_{l,k} = \dfrac{\sin\Omega(t_k - t_l)}{\pi(t_k - t_l)}$ and $\mathbf{E}_\text{t}\,(K\text{x}1)$ - $\dfrac{\sin\Omega(t - t_l)}{\pi(t - t_l)}$.

Uniformly sampled signal $x(kT)$ can be considered as a special case of nonuniform sampling at time moments t_k=kT, k=0,1,2,…,K-1. Then the matrices elements in (17.1-17.3) are

$\mathbf{X}\,(1\text{x}K)$ - $x(kT)$, $\mathbf{E}_\omega\,(K\text{x}1)$ - $e^{-j\omega lT}$, $\mathbf{R}\,(K\text{x}K)$ - $r_{l,k} = \dfrac{\sin\Omega T(k-l)}{\pi T(k-l)}$, $\mathbf{E}_\text{t}\,(K\text{x}1)$ - $\dfrac{\sin\Omega(t - lT)}{\pi(t - lT)}$.

In particular, if uniform sampling of signal $x(kT)$ is done with Nyquist rate $T=\pi/\Omega$, then the matrix **R** becomes a unit matrix and formula (17.1) coincide with classical DTFT (3.3). In case, mean sampling period is less then it demands by Nyquist criteria for uniformly sampled signal, the EDTFT approach can provide a high frequency resolution and improved spectral estimation quality. Unfortunate an achievement of such results is limited by finite precision in the mathematical calculations and by restrictions on frequency range in the process of signal sampling. Theoretical value of denominator in (17.3) $\mathbf{E}_\omega^H\mathbf{R}^{-1}\mathbf{E}_\omega = K$ and the frequency resolution should increase proportionally to the number of samples in the signal observation period Θ. In the border-case, if number of samples within Θ increasing infinitely ($K\rightarrow\infty$) and the discrete time signal tends to the continuous time signal $x(t)$, the EDTFT (17.1) gives the same results as (14.1).

Extended DTFT

Now, let consider the solution of the minimum least square error estimators (9.2) and (9.3) for arbitrary selected signal model $S(\omega_0)$ (see **[2]**, **[3]** and **[5]**). The derivation formulas for both estimators are similar to ones given in previous section. For example, a partial derivation of (9.2) by basis functions $\dfrac{\partial\Delta}{\partial\alpha(\omega, t_l)} = 0$, for $l = 0,1,2,...,K-1$ provide the least square solution

$$\int_{-\Omega}^{\Omega} \left(2\pi S(\omega_0)\delta(\omega-\omega_0) - \sum_{k=0}^{K-1} S(\omega_0)e^{j\omega_0 t_k}\alpha(\omega, t_k) \right) S^*(\omega_0)e^{-j\omega_0 t_l}d\omega_0 = 0 , \tag{16}$$

where (*) denote the complex conjugate value. Equation (16) can be written as

$$\sum_{k=0}^{K-1} \left(\int_{-\Omega}^{\Omega} \left|S(\omega_0)\right|^2 e^{j\omega_0(t_k - t_l)}d\omega_0 \right) \alpha(\omega, t_k) = 2\pi \int_{-\Omega}^{\Omega} \left|S(\omega_0)\right|^2 e^{-j\omega_0 t_l}\delta(\omega-\omega_0)d\omega_0 . \tag{17}$$

The filtering feature of Dirac delta function $\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0)$ applied to the right part of (17) gives the final form of the system of linear equations

$$\sum_{k=0}^{K-1}\left(\frac{1}{2\pi}\int_{-\Omega}^{\Omega}|S(\omega_0)|^2 e^{j\omega_0(t_k-t_l)}d\omega_0\right)\alpha(\omega,t_k) = |S(\omega)|^2 e^{-j\omega t_l}, \qquad (18.1)$$

$$\sum_{k=0}^{K-1}\left(\frac{1}{2\pi}\int_{-\Omega}^{\Omega}|S(\omega_0)|^2 e^{j\omega_0 T(k-l)}d\omega_0\right)\alpha(\omega,kT) = |S(\omega)|^2 e^{-j\omega lT}, \qquad (18.2)$$

where $|S(\omega)|^2$ is the signal model power spectrum at $\omega_0=\omega$. Note that (18.2) is written for uniformly sampled signal $x(kT)$ and can be derived from (9.3) in a similar way. The EDTFT basis functions $\mathbf{A}_\omega$ (Kx1) - $\alpha(\omega,t_k)$ or $\alpha(\omega,kT)$ are found as a solution of (18)

$$\mathbf{A}_\omega = |S(\omega)|^2 \mathbf{R}^{-1}\mathbf{E}_\omega. \qquad (19)$$

Substituting of transform basis (19) into expressions (5) and (10), yields the formulas for calculation of the EDTFT:

$$F_\alpha(\omega) = |S(\omega)|^2 \mathbf{X}\mathbf{R}^{-1}\mathbf{E}_\omega,\ -\Omega \le \omega \le \Omega, \qquad (20.1)$$

$$x_\alpha(t) = \mathbf{X}\mathbf{R}^{-1}\mathbf{E}_t,\ -\infty < t < \infty, \qquad (20.2)$$

$$S_\alpha(\omega) = \frac{\mathbf{X}\mathbf{R}^{-1}\mathbf{E}_\omega}{\mathbf{E}_\omega^H\mathbf{R}^{-1}\mathbf{E}_\omega}. \qquad (20.3)$$

The elements of matrices **R** and $\mathbf{E}_t$ in the formulas (19, 20.1-20.3) are expressed by integrals

R (KxK) - $r_{l,k} = \frac{1}{2\pi}\int_{-\Omega}^{\Omega}|S(\omega_0)|^2 e^{j\omega_0(t_k-t_l)}d\omega_0$, $\mathbf{E}_t$ (Kx1) - $\frac{1}{2\pi}\int_{-\Omega}^{\Omega}|S(\omega)|^2 e^{j\omega(t-t_l)}d\omega$,

for nonuniformly sampled signal **X** (1xK) – $x(t_k)$, and

R (KxK) - $r_{l,k} = \frac{1}{2\pi}\int_{-\Omega}^{\Omega}|S(\omega_0)|^2 e^{j\omega_0(k-l)T}d\omega_0$, $\mathbf{E}_t$ Kx1) - $\frac{1}{2\pi}\int_{-\Omega}^{\Omega}|S(\omega)|^2 e^{j\omega(t-lT)}d\omega$,

for case of uniformly sampled signal **X** (1xK) – $x(kT)$, where k,l=0,1,2,…,K-1.

The frequency resolution of the EDTFT is in inverse ration to $|S(\omega)|^2\mathbf{E}_\omega^H\mathbf{R}^{-1}\mathbf{E}_\omega$ and varied in the frequency range -Ω≤ω≤Ω.

The EDTFT, where the signal model amplitude spectrum $S(\omega_0)$ tends to the signal amplitude spectrum $S_\alpha(\omega)$, can be found by the following iterative algorithm:

Iteration 1. Calculate $S_a^{(1)}(\omega)$ (17.3) applying default signal model $S(\omega_0)$=1.

Iteration 2. Calculate $S_a^{(2)}(\omega)$ (20.3) by using the signal model $S_a^{(1)}(\omega_0)$.

Iteration 3. Calculate $S_a^{(3)}(\omega)$ (20.3) by using the signal model $S_a^{(2)}(\omega_0)$.

...

Iteration i. Calculate $S_a^{(i)}(\omega)$ (20.3) by using the signal model $S_a^{(i-1)}(\omega_0)$.

Iterations are repeated until $S_a^{(i)}(\omega) \approx S_a^{(i-1)}(\omega)$. The EDTFT output $F_a^{(I)}(\omega)$ (20.1) is calculated for the last performed iteration I. The signal model spectrum $S(\omega_0) = S_a^{(I)}(\omega_0)$ and the matrix **R** represent a signal autocorrelation matrix. By default the signal model $S(\omega_0)$=1 is used as input of the EDTFT algorithm. At the same time, an additional information about the signal to be analyzed can be used to create a more realistic signal model for the EDTFT input. This will reduce the number of iterations needed to reach the stopping iteration criteria.

Extended DFT algorithm

The EDTFT considered in previous sections is a function of continuous frequency ($-\Omega \le \omega \le \Omega$), while described below the Extended DFT (EDFT) algorithm calculated the EDTFT on a discrete frequency set $-\Omega \le 2\pi f_n < \Omega$, $n=0,1,2,\ldots,N-1$. The number of frequency points N should be selected sufficiently great to substitute the integrals in expressions (20.1, 20.2) by the finite sums - $N \ge K$, where K is the length of nonuniformly or uniformly sampled sequence $\mathbf{X}$ (1xK). The EDFT can be expressed by the following iterative algorithm

$$\mathbf{R} = \frac{1}{N}\mathbf{E}\mathbf{W}^{(i)}\mathbf{E}^{H}, \tag{21.1}$$

$$\mathbf{F}^{(i)} = \mathbf{X}\mathbf{R}^{-1}\mathbf{E}\mathbf{W}^{(i)}, \tag{21.2}$$

$$\mathbf{S}^{(i)} = \frac{\mathbf{X}\mathbf{R}^{-1}\mathbf{E}}{diag(\mathbf{E}^{H}\mathbf{R}^{-1}\mathbf{E})}, \tag{21.3}$$

where the iteration number $i=1,2,3,\ldots I$. The diagonal weight matrix $\mathbf{W}^{(i)}$ (NxN) for the first iteration is a unit matrix, $\mathbf{W}^{(1)}=\mathbf{I}$, and for the next iterations are derived from the amplitude spectrum $\mathbf{W}^{(i+1)} = diag\,|\mathbf{S}^{(i)}|^{2}$. The matrix $\mathbf{E}$ (KxN) has elements $e^{-j2\pi f_n t_k}$. The $diag(\mathbf{E}^{H}\mathbf{R}^{-1}\mathbf{E})$ (1xN) means extracting the main diagonal elements from quadratic matrix. The EDFT output $\mathbf{F}$ (1xN) and $\mathbf{S}$ (1xN) are calculated from the results of the last performed iteration I. Reconstructed sequence $\mathbf{X}_\alpha$ (1xN) - $x_\alpha(t_m)$, $m=0,1,2,\ldots,N-1$, is obtained by applying an inverse DFT of (21.2)

$$\mathbf{X}_\alpha = \frac{1}{N}\mathbf{F}\mathbf{E}_t, \tag{22}$$

where $\mathbf{E}_t$ (NxN) - $e^{j2\pi f_n t_m}$. Note that $x_\alpha(t_m)=x(t_k)$, if $t_m=t_k$. For $N>K$, the reconstructed by the formula (22) sequence is the original sequence plus forward and backward extrapolation of $\mathbf{X}$ to length N and/or interpolation if there are gaps inside of $\mathbf{X}$. The maximum frequency resolution of the EDFT algorithm is limited by the length N of frequency set, not by the length K of sequence $\mathbf{X}$ as in application of classical DFT. This mean, the EDFT is able to increase the frequency resolution N/K times in comparison with classical DFT. At the same time, there is a restriction on the frequency resolution $sum(\mathbf{F}^{(i)}/\mathbf{S}^{(i)}) = NK$, satisfied by each iteration, and in order to achieve a high resolution on some frequencies, the EDFT should decrease the resolution on other frequencies. In a border-case $N=K$, the EDFT output do not depend on weight matrix $\mathbf{W}$ and can be calculated in a non-iterative way (as result of the first EDFT iteration).

Computer simulations

The computer modeling results are presented for the test signal consisting of three non-overlapping components symbolized in Fig. 1. The sequences $x(kT)$ and $x(t_k)$ ($T_s=T$) length $K=64$ are derived by simulating 8-bit ADC (analog-to-digital converter) of test signal $x(t)$.

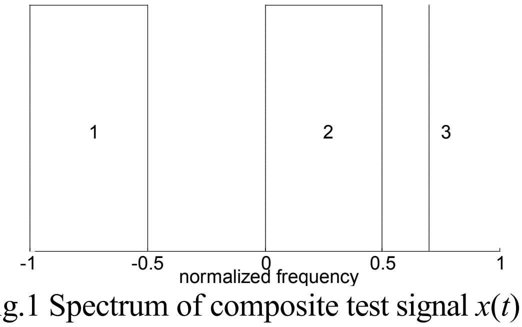

Fig.1 Spectrum of composite test signal $x(t)$:
1 - a band-limited noise, 2 - the rectangular impulse, 3 - complex exponent at $\omega/\Omega=0.7$.

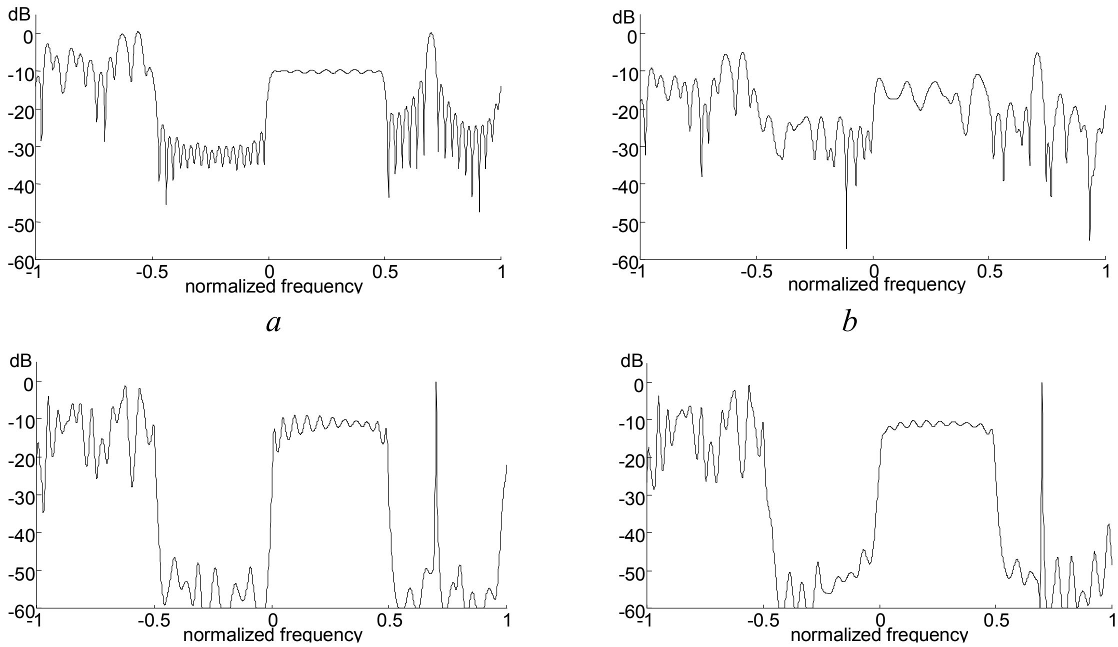

Fig.2 The power spectrum estimation for sampled test signal:
a,b - DFT*, c,d* - EDFT (*a,c* - uniform, *b,d* - nonuniform sampling, T_s=T).

Fig.2 show the results of EDFT 10th iteration in comparison with classical DFT (formulas 3.2 and 3.3). EDFT output (c,d) illustrate, that the proposed algorithm providing a high-frequency resolution, is able to estimate a composite signal spectrum, and is working equally well for uniformly and nonuniformly sampled signal.

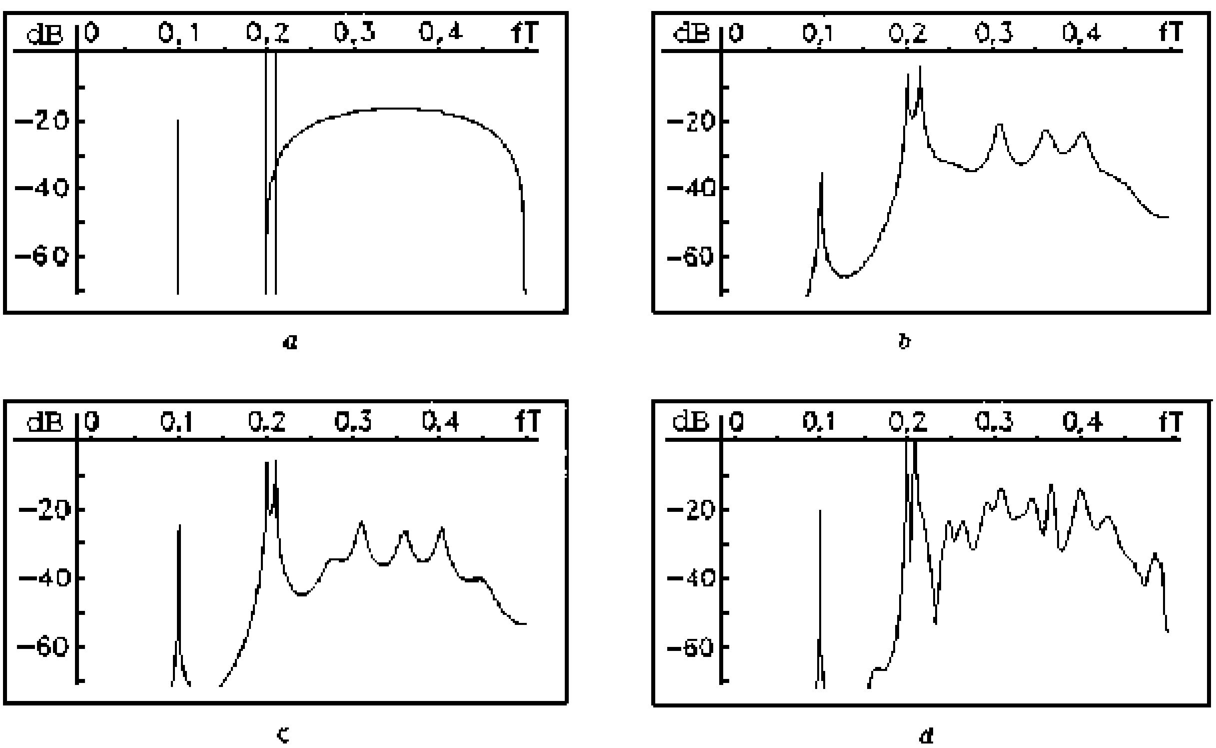

Fig.3 The spectrum estimation of Marple&Kay test signal:
a - true spectra, b - Burg method, c - modified covariance method, d - EDFT.

Fig.3 allow to compare the EDFT 10th iteration output with well-known Burg and modified covariance methods (model order = 16) **[2]**. Fig.3a illustrates the true spectrum of the test signal used by Marple&Kay - 64-point real sample sequence from a process consisting of three sinusoids and a colored noise (see S.M. Kay, S.L. Marple, Spectrum analysis - a modern perspective, Proc. IEEE, No.11, Vol.69, 1981.). Note that the EDFT correctly estimate amplitude and phase spectrum of the test signal. Thus, application of the formula (22), allow to reconstruct not only initial test signal, but also extrapolate it outside the given time interval. Comprehensive computer simulation results of proposed EDFT algorithm and comparison with other spectral analysis methods are given in **[5]**.

EDFT algorithm in MATLAB code

The program NEDFT.m in MATLAB code is created to demonstrate the EDFT algorithm capabilities, which are described in previous sections. The program can be run for nonuniformly or uniformly sampled signals and for arbitrary selected frequency set f_n (see NEDFT.m program help for details). From the calculations complexity viewpoint, it is reasonable to select the frequencies on the same grid as used by the Fast Fourier Transform (FFT) algorithm. The program EDFT.m is designed as a faster realization of the EDFT algorithm **[4]**. This program is applicable for uniformly sampled signals and for signals with missing samples or data segments (gaps) inside of the input sequence (see EDFT.m program help for more details). The first version of the EDFT (file GDFT.m) was submitted on 10/7/1997 as MATLAB 4.1 code. The renewed EDFT version submitted on 8/5/2006 and available online http://www.mathworks.com/matlabcentral/fileexchange/11020-extended-dft. Note that programs have not been tested under latest MATLAB versions, and therefore have a lot of space for performance improvements.

Selected reference articles

[1] V. Liepin'sh, A method for spectral analysis of band-limited signals, Automatic control and computer sciences /*available in Russian and English*/, Vol. 27, No. 5, pp. 56-63, 1993.

[2] V. Liepin'sh, A method for spectrum evaluation applicable to analysis of periodically and non-regularly digitized signals, Automatic control and computer sciences, Vol. 27, No. 6, pp. 57-64, 1993.

[3] V. Liepin'sh, A spectral estimation method of nonuniformly sampled band-limited signals, Automatic control and computer sciences, Vol. 28, No. 2, pp. 66-73, 1994.

[4] V. Liepin'sh, An algorithm for evaluation a discrete Fourier transform for incomplete data, Automatic control and computer sciences, Vol. 30, No. 3, pp. 27-40, 1996.

[5] Vilnis Liepins, High-resolution spectral analysis by using basis function adaptation approach /*in Latvian*/, Doctoral Thesis for Scientific Degree of Dr.Sc.Comp., University of Latvia, 1997. Abstract available on http://hdl.handle.net/10068/330816 .

Acknowledgments

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http://www.springerlink.com/content/kj8w474v677n6880/
http://d.wanfangdata.com.cn/Periodical_dlxtjqzdhxb201003002.aspx
http://www.eurasip.org/Proceedings/Eusipco/Eusipco2010/Contents/papers/1569292079.pdf
http://www.meps10.pwr.wroc.pl/submission/data/papers/16.1.pdf

http://sisl.seas.harvard.edu/files/pub/BarbedoLopesWolfe08.pdf
http://ieeexplore.ieee.org/xpl/freeabs_all.jsp?tp=&arnumber=4477348&isnumber=4626339
http://ismir2007.ismir.net/proceedings/ISMIR2007_p399_barbedo.pdf
http://www.edi.lv/media/uploads/UserFiles/dasp-web/dasp-papers/Greitans_sampta01.pdf